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ON THE TORUS THEOREM FOR CLOSED 3-MANIFOLDS

BY

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ABSTRACT. In this paper we give the appropriate generalization of the torus theorem to closed, sufficiently large, irreducible, orientable 3-manifolds.

I. Introduction. In [2] we proved the torus theorem for a bounded, orientable, 3-manifold M and conjectured that the theorem would also hold if M were sufficiently large, closed, irreducible, and orientable. W. Jaco has pointed out to the author that one can construct a counterexample to our conjecture by sewing a solid torus M_1 to the knot space M_2 of a torus knot so that the fibrings of M_1 and M_2 agree. Of course one requires that the spanning surface of the torus knot is of genus greater than one.

It is the purpose of this paper to prove that if M is a sufficiently large, closed, orientable irreducible 3-manifold that admits an essential map of a torus, either M admits an essential embedding of a torus or a finite sheeted covering space of M has a particular structure. It would be interesting if all closed, orientable irreducible 3-manifolds that admit essential maps of tori and not essential embeddings of tori have covering spaces with this structure. In particular such 3-manifolds would be "almost sufficiently large."

Theorem 1 also aims at a partial answer to question 3 in [4] for genus 1 surfaces.

The results of this paper also follow from theorems proved independently by Johanssen and by Jaco and Shalen, which classify the boundary-preserving maps of a torus or annulus into a sufficiently large 3-manifold, up to boundary-preserving homotopy.

II. Notation. We adopt the notation and conventions in [7] without change. We let A represent an annulus, c_1 and c_2 the components of ∂A and α a spanning arc of A (i.e. $A - \alpha$ is connected and simply connected) throughout this paper. Let M be a 3-manifold and F an incompressible surface in ∂M. Then $f\colon (A, \partial A) \longrightarrow (M, F)$ is an *F-essential map* if

(1) $f_*\colon \pi_1(A) \longrightarrow \pi_1(M)$ is monic.

(2) $f(\alpha)$ is not homotopic rel its boundary to an arc on F.

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If $F = \partial M$, we say that f is *essential*. Let T be a torus. A map $f\colon T \longrightarrow M$ is *essential* if

(1) $f_*\colon \pi_1(T) \longrightarrow \pi_1(M)$ is monic.

(2) There is a loop $\lambda \subset T$ such that $f(\lambda)$ is not freely homotopic to a loop in ∂M. Note that the second condition does not apply if M is closed.

Let F be a surface embedded in M or ∂M. Let $g_1, g_2\colon (A, \partial A) \longrightarrow (M, F)$, then g_1 and g_2 are *parallel* rel F if there is an embedding $H\colon A \times I \longrightarrow M$ such that

(1) $H(A \times \{0\}) = g_1(A)$,

(2) $H(A \times \{1\}) = g_2(A)$,

(3) $H(\partial A \times I) \subseteq F$.

Let M_1 be a 3-submanifold of M. Then $\mathrm{Fr}(M_1, M)$, or simply $\mathrm{Fr}(M_1)$ when no confusion can result, is the closure of $(\partial M_1 \cap (M - \partial M))$.

III. **Supporting results.** The following results are useful in the proof of the principal theorem of this paper. Proposition 4 is of some interest in itself.

LEMMA 1. *Let M be a closed, connected, irreducible 3-manifold and F a closed, connected, 2-sided, incompressible surface of minimal genus embedded in M. Let N be a regular neighborhood of F in M. Let $f\colon T \longrightarrow M$ be an essential map. Suppose*

(1) *$f^{-1}(F)$ is the union of a nonempty collection of essential simple loops.*

(2) *f is not homotopic to a map $\bar{f}\colon T \longrightarrow M$ such that $\bar{f}^{-1}$ contains fewer components than $f^{-1}(F)$.*

(3) *The f image of each component of $T - f^{-1}(F)$ meets both components of $N - F$.*

Then there is an essential map $g\colon K \longrightarrow M$ where K is either a torus or a Klein bottle such that

(1) *$g^{-1}(F)$ is the union of a nonempty collection of essential simple loops.*

(2) *$K - g^{-1}(F)$ is the union of a collection of open annuli.*

(3) *The restriction of g to the closure of each component of $K - g^{-1}(F)$ is not homotopic* rel *$g^{-1}(F)$ to a map into F.*

(4) *The restriction of g to each component of $g^{-1}(F)$ and $K - g^{-1}(F)$ is a homeomorphism.*

PROOF. This is Lemma 5.11 in [2] except that M is a closed manifold.

LEMMA 2. *Let F be a closed, connected, 2-sided, incompressible surface of minimal genus properly embedded in M. Let N be a regular neighborhood of F in M. Let K_1 be a torus or a Klein bottle and $f\colon K_1 \longrightarrow M$ an essential map such that*

(1) *$f^{-1}(F)$ is the union of a nonempty collection of disjoint simple loops.*

(2) *The components of* $K_1 - f^{-1}(F)$ *are open annuli whose closures we denote by* $A_1, \ldots, A_n$.

(3) $f(A_1)$ *meets only one component of* $R - F_1$.

(4) $f | A_i$ *is not homotopic* rel ∂A_i *to a map into* F *for* $i = 1, \ldots, n$.

(5) f *is not homotopic to a map* f_1 *such that* $f_1^{-1}(F)$ *contains fewer loops than* $f^{-1}(F)$.

Then there is an essential embedding $g: T \longrightarrow M$.

PROOF. This is an immediate consequence of Lemma 5.12 in [2] since the boundary of a regular neighborhood of an essential embedding of a Klein bottle in M is an essential torus.

LEMMA 3. *Let* M *be a closed, irreducible, orientable* 3-*manifold and* F *a closed, connected,* 2-*sided, nonseparating, incompressible surface embedded in* M. *If* M *admits an essential map* $f: T \longrightarrow M$, *either*

(1) M *admits an essential embedding* $g: T \longrightarrow M$ *or*

(2) M *is a fibre space with base* S^1 *and fibre* F *and there is a finite sheeted cyclic cover* $(\widetilde{M}, p)$ *of* M *associated with* F *such that* $\widetilde{M}$ *is homeomorphic to* $F \times S^1$.

PROOF. After the usual argument, we suppose that $f^{-1}(F)$ is the union of a collection of disjoint, essential, simple loops. If $f^{-1}(F)$ is empty, it is a consequence of Theorem 8 in [2] that there is an embedding $g: T \longrightarrow M - F$ such that $g_*: \pi_1(T) \longrightarrow \pi_1(M - F)$ is an injection. Since $\pi_1(M - F) \longrightarrow \pi_1(M)$ is monic, $g: T \longrightarrow M$ is an essential map. Thus we may suppose that $f^{-1}(F)$ is the union of a nonempty collection of disjoint simple essential loops $\lambda_1, \ldots, \lambda_n$ and that the number of loops in this collection cannot be reduced by a homotopy of f.

Let N be a regular neighborhood of F in M. If the f image of some component of $T - f^{-1}(F)$ meets both components of $N - F$, Lemma 3 is an immediate consequence of Lemma 2. Thus we may suppose that the f image of each component of $T - f^{-1}(F)$ meets both components of $N - F$. It is now a consequence of Lemma 1 that there is an essential map $f: K \longrightarrow M$ where K is either a torus or a Klein bottle such that

(1) $f^{-1}(F)$ is the union of a nonempty collection of disjoint simple essential loops.

(2) Each component of $K - f_1^{-1}(F)$ is an open annulus.

(3) The restriction of f_1 to each component of $K - f_1^{-1}(F)$ and $f_1^{-1}(F)$ is a homeomorphism.

(4) f_1 is not homotopic to a map $\bar{f}_1$ such that $\bar{f}_1^{-1}(F)$ has fewer components than $f_1^{-1}(F)$.

If the f_1 image of some component of $K - f_1^{-1}(F)$ meets only one component of $N - F$, we apply Lemma 2 to complete the proof of Lemma 3. If $f_1^{-1}(F)$ is a single loop, f_1 is an embedding and we are finished. Let $A_1, \ldots, A_n$ be the closures of the components of $K - f_1^{-1}(F)$ and $\lambda_1, \ldots, \lambda_n$ the components of $f_1^{-1}(F)$. We may now suppose that $f_1(A_i - \partial A_i)$ meets both components of $N - F$ for $i = 1, \ldots, n$ where $n \geqslant 2$.

We suppose that f has been chosen so that the number of loops in $f^{-1}(F)$ is minimal. Suppose $A_i \cap f^{-1}(A_j)$ contains an essential simple loop λ where $1 \leqslant i < j \leqslant n$. Let $\lambda_2 \subset A_j$ be a loop on K such that $f(\lambda_2) = f(\lambda_1)$. Since λ_1 is essential, λ_2 is essential. Now $K - (\lambda_1 \cup \lambda_2)$ is the union of two open annuli whose closures we denote by B_1 and B_2. We observe that $f(B_1)$ defines an essential map $\bar{f}$ of a closed connected surface K^1 of Euler characteristic zero such that $\bar{f}(K^1) = f(B_1)$, $\bar{f}$ is essential by Lemma 5.3 in [2]. This is easily seen to contradict our assumption that f was chosen so that $f^{-1}(F)$ would contain a minimal number of loops. Thus $A_i \cap f^{-1}(A_j)$ contains no essential simple loops for $1 \leqslant i, j \leqslant n$ and $i \neq j$.

For any essential map $\bar{f}: K \longrightarrow M$ such that $\bar{f}^{-1}(F) = \bigcup_{i=1}^{n} \lambda_i$, we define $X(\bar{f}) = \bigcup_{i \neq j}^{n} \bar{f}(\lambda_i) \cap \bar{f}(\lambda_j)$. We suppose that f_1 has been chosen to satisfy (1)–(5) below and so that there is no map $\bar{f}$ homotopic to f_1 such that

(1) $\bar{f}^{-1}(F) = f_1^{-1}(F) = \bigcup_{i=1}^{n} \lambda_i$.

(2) $\bar{f} \mid \lambda_i$ is a homeomorphism for $i = 1, \ldots, n$.

(3) $\bar{f} \mid (A_i - \partial A_i)$ is a homeomorphism for $i = 1, \ldots, n$.

(4) The cardinality of $X(\bar{f})$ is less than that of $X(f_1)$.

(5) $\bar{f}(\lambda_i) \cap \bar{f}(\lambda_j) \cap \bar{f}(\lambda_k) = \varnothing$ if $1 \leqslant i < j < k \leqslant n$.

Now it can be seen that $X(f_1)$ is a finite set. We may assume that

$$J_{ij} = \mathrm{cl}(A_i \cap f_1^{-1} f_1(\mathrm{int}(A_j)))$$

is the union of a collection of disjoint simple loops and arcs properly embedded in A_i where $1 \leqslant i < j \leqslant n$.

We claim every arc in J_{ij} properly embedded in A_i is a spanning arc of A_i. Suppose $\beta_1 \subseteq J_{ij}$ is an arc properly embedded in A_i such that $\partial\beta_1$ lies on a single component of ∂A_i. Now β_1 cuts off a disk $\mathcal{D}_1 \subseteq A_i$. By construction $f_1 \mid \mathcal{D}_1$ is a homeomorphism. Let β_2 be the arc on A_j such that $f_1(\beta_2) = f_1(\beta_1)$. Now β_2 cuts off a disk $\mathcal{D}_2 \subseteq A_j$ and $f_1 \mid \mathcal{D}_2$ is a homeomorphism. We may choose $\mathcal{D}_1$ so that $f_1(\mathcal{D}_1) \cap f_1(\mathcal{D}_2)$ is the union of $f_1(\beta_1)$ and a collection (possibly empty) of disjoint simple loops properly embedded in $f_1(\mathcal{D}_1)$. Let β_1' and β_2' be the closures of $\partial\mathcal{D}_1 - \beta_1$ and $\partial\mathcal{D}_2 - \beta_2$ respectively. Now $f_1(\beta_1') \cup f_2(\beta_2')$ is a simple loop $\lambda \subset \bigcup_{i=1}^{n} f_1(\lambda_i)$ and λ is nullhomotopic in M across the singular disk $f_1(\mathcal{D}_1) \cup f_1(\mathcal{D}_2)$. Since F is incompressible, we may suppose that λ bounds a disk $\mathcal{D}$ embedded in F. After the usual argument, we may suppose that each arc $\beta \subset$

$\bigcup_{i=1}^n f_1(\lambda_i)$ properly embedded in $\mathcal{D}$ meets $f_1(\beta_j')$ in a single point for $j = 1, 2$. Since $f_1(\beta_1')$ is isotopic rel its boundary to $f_1(\beta_2')$ across $\mathcal{D}$ it can be seen that f_1 was not chosen so that $X(f_1)$ would be minimal.

Similarly if β_1 and β_2 are arcs in λ_i and λ_j where $1 \leq i < j \leq n$ and $f_1(\beta_1) \cup f_1(\beta_2)$ bounds a disk $\mathcal{D} \subseteq F$, we can show that the best possible choice was not made for f_1.

Let N_1' be a regular neighborhood of $\bigcup_{i=1}^n f_1(\lambda_i)$ in F. Now N_1' is a 2-submanifold of F. Let $N_1 \subseteq F$ be the smallest submanifold of F that contains some given component of N_1' so that $\pi_1(N_1) \longrightarrow \pi_1(F)$ is an injection. We claim that there is a submanifold M_1 of M such that

(1) M_1 is a fibre bundle with base S^1 and fibre N_1.

(2) N_1 is a fibre of M_1.

(3) $\pi_1(M_1) \longrightarrow \pi_1(M)$ is an injection.

In this case if N_1 has boundary, a component of ∂M_1 will be our desired essential embedding. Otherwise M is a fibre bundle with base S^1 and fibre F.

Let M^* be the manifold obtained by splitting M along F and P: $M^* \longrightarrow M$ the natural projection map. We will assume that N_1' is connected; however, the interested reader will easily be able to fill in the missing details in the general case. Let Q_1 and Q_2 be the components of $P^{-1}(N_1)$. Clearly we need only show that there is an embedding of H: $N_1 \times I \longrightarrow M^*$ such that $H(N_1 \times \{0\}) = Q_1$, $H(N_1 \times \{1\}) = Q_2$ and $PH(N_1 \times I) = M_1$.

Let h_i: $A_i \longrightarrow M^*$ be the map induced by $f_1|A_i$ for $i = 1, \ldots, n$. Let $\mu_i = h_i(A_i) \cap Q_1$ and $\mu_i' = h_i(A_i) \cap Q_2$ for $i = 1, \ldots, n$. We may suppose that $\bigcup_{j=1}^i \mu_j \cap \mu_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$ since N_1 has been taken to be connected. Let R_1 be a regular neighborhood of $h_1(A_1)$. Now h_2 is homotopic to an embedding h_2^* rel ∂A_2 so that the closure of $(\partial R_1 - (Q_1 \cup Q_2)) \cap h_2^*(A_2)$ is the union of a nonempty collection of disjoint simple arcs and loops. Since $\pi_2(M^*) = 0$ as a consequence of the irreducibility of M and the sphere theorem [4], [8], $\partial R_1 \cap h_2^*(A_2)$ may be taken to be a collection of disjoint simple arcs. We observe that each component γ of $Q_1 \cap R_1 \cap h_2^*(\partial A_2)$ contains a crossing point of $h_1(\partial A_1)$ and $h_2(\partial A_2)$. Thus the component $\bar{\mathcal{D}}$ of $h_2^*(A_2) \cap R_1$ containing γ contains an arc running from Q_1 to Q_2. It is easily seen that $\bar{\mathcal{D}}$ is either a disk or all of $h_2^*(A_2)$ and that $\bar{\mathcal{D}} \cap (R_1 - (Q_1 \cup Q_2))$ is a pair of disjoint simple arcs running from Q_1 to Q_2. Let $\bar{R}_2$ be a regular neighborhood of $R_1 \cup h_2^*(A_2)$. It can be seen that $\bar{R}_2$ is homeomorphic to the product of $F_2 = \bar{R}_2 \cap Q_1$ with the unit interval. If $\pi_1(F_2) \longrightarrow \pi_1(Q_1)$ is not an injection, there is a disk E_1 embedded in Q_1 such that $E_1 \cap F_2 = \partial E_1$. Now ∂E_1 is freely homotopic in $\partial \bar{R}_2$ to a loop in Q_2 so there is a disk E_2 embedded in Q_2 such that $E_2 \cap \partial \bar{R}_2 = \partial E_2$. Now $E_1 \cup E_2$ together with an annulus in $\partial \bar{R}_2$ is a 2-sphere that bounds a 3-ball B in M^* and $B \cup \bar{R}_2$ is homeomorphic to the product of $F_2 \cup E_1$ with the unit interval. We let R_2 be the smallest submanifold of M^* such that

(1) $R_2 \supseteq \bar{R}_2$.

(2) R_2 is homeomorphic to the product of $R_2 \cap Q_1$ with the unit interval under a homeomorphism which carries $R_2 \cap Q_1$ to $(R_2 \cap Q_1) \times \{0\}$ and $(R_2 \cap Q_2)$ to $(R_2 \cap Q_1) \times \{1\}$.

(3) $\pi_1(R_2 \cap Q_1) \longrightarrow \pi_1(Q_1)$ is an injection.

We admit that the proof above is more complicated than is necessary to find R_2, but the proof above also suffices to extend R_2 to R_3 when h_3^* is chosen to be a map homotopic to h_3 rel ∂A_3 such that the closure of $(\partial R_2 - (Q_1 \cap Q_2)) \cap h_3^*(A_3)$ is the union of a nonempty collection of disjoint simple arcs each of which runs from Q_1 to Q_2 and our claim follows after an inductive argument.

We assume that M is a fibre space with base S^1 and fibre F and observe that any finite sheeted covering associated with F will also be such a fibre space. It follows from the proof above that $\pi_1(f(T) \cap F) \longrightarrow \pi_1(F)$ is an epimorphism.

Let $(\widetilde{M}, p)$ be the n-sheeted cyclic cover of M associated with F. Let $\widetilde{f}$: $K \longrightarrow \widetilde{M}$ be a lift of f. Note that $\widetilde{f}$ is an embedding. Let ρ: $\widetilde{M} \longrightarrow \widetilde{M}$ be a generator of the group of covering translations of $\widetilde{M}$. Note that $\rho^i(K) \cap \rho^j(K)$ is the union of a collection of disjoint simple loops for $0 \leqslant i < j < n$. Let m be twice the least common multiple of the intersection numbers of loops in $\rho^i(K) \cap \rho^j(K)$ and F for $1 \leqslant i < j \leqslant n - 1$. Let $(M^\#, q)$ be the m-sheeted cyclic cover of $\widetilde{M}$ associated with any component of $\rho^{-1}(F)$.

We claim that $M^\#$ is homeomorphic to $F \times S^1$. Observe that $q^{-1}(\rho^i\widetilde{f}(K))$ is a torus embedded in M^* for $0 \leqslant i < n$ and that each essential loop in $q^{-1}(\rho^i\widetilde{f}(K)) \cap q^{-1}(\rho^j\widetilde{f}(K))$ meets each component of $(pq)^{-1}(F)$ in either the empty set or a singleton set.

Let $F^\#$ be a component of $(pq)^{-1}(F)$. We split $M^\#$ along $F^\#$ to obtain a 3-manifold $\overline{M}$ and let P: $\overline{M} \longrightarrow M^\#$ be the natural projection map. Now $q^{-1}(\rho^i\widetilde{f}(K))$ induces an embedding

$$h_i\colon (A, \partial A) \longrightarrow (\overline{M}, P^{-1}(F^\#)) \quad \text{for } 0 \leqslant i < n.$$

By construction $h_i(A) \cap h_j(A)$ is the union of a collection of disjoint simple spanning arcs and inessential simple loops for $0 \leqslant i < j < n$. Let $\overline{F}_1$ and $\overline{F}_2$ be the components of $P^{-1}(F^\#)$. We may suppose that $h_i(c_1) \subseteq \overline{F}_1$ for $0 \leqslant i < n$ and that $\bigcup_{i=1}^k h_i(c_1) \cap h_{k+1}(c_1)$ is not empty for $1 \leqslant k < n - 1$ since we have shown that $f(K) \cap F$ is connected or M admits an essential embedding of T.

Let $\bar{\alpha} \subset h_1(A) \cap h_2(A)$ be a spanning arc of $h_1(A)$. Observe that $P(\partial\alpha)$ is a single point. Let h_2^* be a map homotopic to h_2 rel ∂A such that $h_2^*(A) \cap h_1(A)$ contains no simple loops. Let N_1 be a regular neighborhood of $h_2^*(c_1) \cup h_1(c_1)$ in $\overline{F}_1$. Then there is an embedding $H_1\colon N_1 \times I \longrightarrow \overline{M}$ such that

(1) $H_1(x, 0) = x$ for $x \in N_1$ and $H_1(N_1 \times \{1\}) \subset \overline{F}_2$.

(2) $PH_1(x, 0) = PH_1(x, 1)$ for $x \in N_1$.

(3) For each arc $\bar{\alpha} \subset h_1(A) \cap h_2^*(A)$, $H_1^{-1}(\alpha) = \{x_0\} \times I$ for some $x_0 \in N_1$.

(4) $H_1^{-1}h_1(A) = h_1(c_1) \times I$.

(5) $H_1^{-1}h_2^*(A) = h_2(c_1) \times I$.

By assumption there is a spanning arc $\bar{\alpha}_1$ of $h_3(A)$ in $h_1(A) \cap h_3(A)$ or in $h_2^*(A) \cap h_3(A)$. We assume the former.

We claim that $H_1^{-1}(\bar{\alpha}_1)$ is homotopic rel its boundary in $H_1^{-1}h_1(A)$ to a product arc in $N_1 \times I$. It follows from our claim that h_3 is homotopic rel ∂A to a map h_3^* such that $h_3^*(A) \cap h_1(A)$ is the union of a collection of disjoint simple spanning arcs and thus that $h_3^*(A) \cap h_2^*(A) \cap h_1(A)$ is empty. In this case we will be able to extend our product structure as was done in proving that M is homeomorphic to a fibre space except that our product structure will be compatible with P so that $\bar{M}$ can be seen to be $F \times S^1$.

It remains to establish our claim. Note that $P(\partial\bar{\alpha}_1)$ is a point so that $H_1^{-1}(\partial\bar{\alpha}_1) = \{x_0\} \times \{0, 1\}$. Let $\theta\colon N_1 \times I \longrightarrow N_1$ be defined by $\theta(x, t) = x$ for $x \in N_1$. Since $PH_1(x, 0) = PH_1(x, 1)$ for $x \in N_1$, $\theta(H_1^{-1}h_3(\partial A))$ contains an arc β such that $H_1(\beta \times \{0, 1\}) \subseteq h_3(\partial A)$ and $H_1(\beta \times \{0\})$ and $H_1(\beta \times \{1\})$ are the components of $h_3(\partial A) \cap H_1(N_1 \times I)$ containing $\partial\bar{\alpha}_1$. Now h_3 is homotopic to a map $h_3^{\#}$ rel $\partial A \cup h_3^{-1}(\bar{\alpha}_1)$ such that $h_3^{\#-1}H_1(\partial N_1 \times I)$ is a collection of disjoint spanning arcs. Let $\mathcal{D}$ be the closure of the component of $A - h_3^{\#-1}H_1(\partial N_1 \times I)$ that contains $h_3^{\#-1}(\bar{\alpha}_1)$. Now $\theta H_1^{-1}h_3^{\#}\colon \mathcal{D} \longrightarrow N_1$ determines a map $\phi\colon (A, \partial A) \longrightarrow (N_1, \partial N_1)$. If $\phi(c_1)$ is nullhomotopic in N_1, $\theta H_1^{-1}(\bar{\alpha}_1)$ is nullhomotopic in N_1 and thus $H_1^{-1}(\bar{\alpha}_1)$ is homotopic to a product arc in $N_1 \times I$. If $\phi(c_1)$ is essential in N_1, either N_1 is an annulus and $h_1(c_1)$ and $h_2(c_2)$ are isotopic in N_1 to disjoint loops or ϕ is homotopic to a map into ∂N_1 and $h_3(c_1)$ is isotopic in $\bar{F}_1$ to a loop not meeting $h_1(c_1) \cup h_2(c_1)$. Either of the above contradicts the minimality of the cardinality of $X(f)$ so our claim is established.

This completes the proof of Lemma 3.

REMARK 1. It is clear from the proof of Lemma 3 that if a manifold M satisfying the conditions of Lemma 3 admits essential embeddings $g_1, g_2\colon T \longrightarrow M$ such that

(1) $g_1(T) \cap F$ and $g_2(T) \cap F$ are unions of collections of disjoint loops.

(2) The number of points in $F \cap g_1(T) \cap g_2(T)$ cannot be reduced by an isotopy of $g_1(T)$ or $g_2(T)$.

(3) $\pi_1((g_1(T) \cup g_2(T)) \cap F) \longrightarrow \pi_1(F)$ is onto (in particular $(g_1(T) \cup g_2(T)) \cap F$ is connected), then M has a finite sheeted covering homeomorphic to $F \times S^1$.

PROPOSITION 4. *Let M be a compact, connected, irreducible 3-manifold* *and F an incompressible surface in ∂M. Let $n \geqslant 1$ and for $i = 1, \ldots, n$ f_i: $(A, \partial A) \longrightarrow (M, F)$ be essential maps. Then there is an embedded line bundle N in M and collection of essential maps $\bar{f}_1, \ldots, \bar{f}_n$: $(A, \partial A) \longrightarrow (M, F)$ such that*

(1) *$\bar{f}_i$ and f_i: $(A, \partial A) \longrightarrow (M, F)$ are homotopic.*

(2) *$N \cap F$ is an incompressible surface in M and is a 2-sheeted (not necessarily connected) cover of the zero section of N.*

(3) *$\partial N \cap \partial F$ contains $\bigcup_{i=1}^{n} \bar{f}_i(\partial A)$.*

(4) *$\mathrm{Fr}(N)$ is the union of a collection of essential annuli in M.*

(5) *No fiber of N is homotopic* rel *its boundary to an arc in F.*

PROOF. Let h_i: $(A, \partial A) \longrightarrow (M, F)$ for $i = 1, \ldots, m$ be a maximal collection of essential embeddings such that $h_i(A) \cap h_j(A)$ is empty for $1 \leqslant i < j \leqslant m$ and $h_i(A)$ and $h_j(A)$ are not parallel rel F for $1 \leqslant i < j \leqslant m$. It is a consequence of Theorem 3 in [1] that this collection is not empty and of the theorem on p. 60 in [7] that the collection is finite. Let N_1 be a regular neighborhood of $\bigcup_{i=1}^{m} h_i(A)$ in M. Then N_1 is homeomorphic to a line bundle. We suppose that the f_i for $i = 1, \ldots, n$ are in general position with respect to $\mathrm{Fr}(N_1)$ and that the number of points in $f_i^{-1}(\partial\, \mathrm{Fr}(N_1))$ cannot be reduced by a homotopy of f_i: $(A, \partial A) \longrightarrow (M, F)$ for $1 \leqslant i \leqslant n$.

Suppose that for some j, where $1 \leqslant j \leqslant n$, f_j is not homotopic to a map $\bar{f}_j$ such that $f_j^{-1}(\partial\, \mathrm{Fr}(N_1))$ is empty. Let $J = f_j^{-1}(\mathrm{Fr}(N_1))$. Now J is the union of a collection of disjoint simple arcs and loops properly embedded in A. Since f_j^*: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic and $\mathrm{Fr}(N_1)$ is incompressible in M, the usual argument shows that J may be assumed to contain no nullhomotopic simple loops.

If J contains an essential loop, each arc β_1 in J that is properly embedded in A has its endpoints on a single component of ∂A. Let $\mathcal{D}$ be the disk on A cut off by β_1. Let β_2 be the closure of $\partial\mathcal{D} - \beta_1$. We observe that $f_j(\beta_1)$ lies on a single component of $\mathrm{Fr}(N_1)$ and since the components of $\mathrm{Fr}(N_1)$ are essential annuli in M, $f_j(\partial\beta_1)$ lies on a single component of $\partial\, \mathrm{Fr}(N_1)$. But now $f_j(\beta_1)$ is homotopic rel its boundary to an arc in $\partial\, \mathrm{Fr}(N_1)$ as is $f_j(\beta_2)$ in F since F is incompressible. It can now be seen that f_j was not chosen so that $f_j^{-1}(\partial\, \mathrm{Fr}(N_1))$ would contain a minimal number of points. Thus J can contain no essential loops.

The argument in the preceding paragraph also shows that each arc $\beta \subseteq J$ and properly embedded in A must be a spanning arc of A. It is not difficult to see that we may suppose that $f_j(\beta)$ is a spanning arc of one of the annuli in $\mathrm{Fr}(N_1)$ and further that $f_j | J$ is an embedding. Note that since $f_j(\partial A) \cap \partial\, \mathrm{Fr}(N_1)$ is not empty, J contains a spanning arc of A.

Let $\mathcal{D}$ be the closure of a component of $A - f_j^{-1}(N_1)$ and N_1^* the closure of $M - N_1$. Observe that $\mathcal{D}$ is a disk and $\partial\mathcal{D} \cap f_j^{-1}(\mathrm{Fr}(N_1))$ is the union of two spanning arcs α_1 and α_2 of A. Let β_1 and β_2 be the closures of the components of $\partial \mathcal{D} - (\alpha_1 \cup \alpha_2)$. If $f_j(\partial \mathcal{D})$ is nullhomotopic in ∂N_1^*, $f_j(\beta_1)$ and $f_j(\beta_2)$ are homotopic in F rel their boundaries to arcs in ∂ Fr(N_1) and the best choice was not made for f_j. It is a consequence of the loop theorem [6] and its proof that there is a disk $\mathcal{D}_1$ properly embedded in N_1^* such that $\partial \mathcal{D}_1$ is essential in ∂N_1^* and $\mathcal{D}_1 \cap \mathrm{Fr}(N_1) \subseteq f_j(\beta_1 \cup \beta_2)$. Since F is incompressible and $\partial \mathcal{D}_1 - \mathrm{Fr}(N_1) \subseteq F$, $\partial \mathcal{D}_1 \cap \mathrm{Fr}(N_1) = f_j(\beta_1 \cup \beta_2)$ and the arcs which are the closures of $\partial \mathcal{D}_1 - \mathrm{Fr}(N_1)$ are not homotopic in F rel their boundaries to arcs in $N_1 \cap F$. Now it can be seen that if $\overline{N}_2$ is a regular neighborhood of $N_1 \cup \mathcal{D}_1$, $\overline{N}_2$ is a (possibly twisted) line bundle.

It is a consequence of an inductive argument and the compactness of F that there is a line bundle $N_2 \subseteq M$ such that

(1) $N_2 \cap F = N_2 \cap \partial M$ is an incompressible surface in M and is a 2-sheeted cover of the zero section of N_2.

(2) $\mathrm{Fr}(N_2)$ is the union of a nonempty collection of essential annuli in M.

(3) If $\overline{f}_i(\partial A) \cap N_2$ is not empty $\overline{f}_i(\partial A) \subseteq N_2$ for where $\overline{f}_i$: $(A, \partial A) \longrightarrow (M, F)$ is homotopic to f_i for $i = 1, \ldots, n$.

(4) No fibre of N_2 is homotopic rel its boundary to an arc in F.

We assume that the $\overline{f}_i$ were originally chosen as the f_i so that we may omit the superscript on f_i when we continue the argument.

Suppose that $f_j(\partial A)$ does not lie in $N_2 \cap F$ and that $f_j(c_1)$ is not homotopic in F to a loop in $N_2 \cap F$. Let F_1 be the closure of the component of $F - N_2$ on which $f_j(c_1)$ lies. If F_1 is not the planar surface with three boundary components, it is a consequence of Theorem 2 in [3] and Theorem 1 in [1] that there is an essential embedding h_{m+1}: $(A, \partial A) \longrightarrow (M, F)$ such that $h_{m+1}(c_1)$ lies on F_1 and $h_{m+1}(c_1)$ is not homotopic in F_1 to a component of ∂F_1. But it would follow that the collection $h_1, \ldots, h_m$ was not maximal. Thus if F_1 is the closure of any component of $F - N_2$ that contains a component of $f_j(\partial A)$, we may suppose that F_1 is the planar surface with three boundary components or an annulus and $f(c_1)$ or $f(c_2)$ is not homotopic in the closure of $F - N_2$ to a loop in $\partial(F \cap N_2)$.

Let M_1 be the closure of a component of $M - N_2$ such that $(M_1 \cap F) \supset F_1 \supset f_j(c_1)$ where F_1 is a planar surface with three boundary components and $f_j(c_1)$ is not homotopic in F_1 to a loop in $N_2 \cap F$. It is a consequence of Theorem 2 in [3], Theorem 1 in [2], and the maximality of the collection $h_1, \ldots, h_m$ that $N_2 \supseteq \partial F_1$.

We claim (a) M_1 is a product line bundle. We show first that ∂M_1 must have total genus two and that $f_j(\partial A)$ does not lie entirely on F_1. Since the Euler characteristic of $F_1 \subseteq \partial M_1$ is -1, it is clear that the component S of ∂M_1 containing F_1 has Euler characteristic at most -2. Suppose $f_j(\partial A) \subseteq F_1$. Let λ be a component of ∂F_1 such that λ does not lie on an annular component of the closure of $S - F_1$. By Theorem 2 in [3] and Theorem 1′ in [1] there is an essential (in M) embedding h_{m+1}: $(A, \partial A) \longrightarrow (M, F_1)$ such that $h_{m+1}(c_1) = \lambda$. Now $h_{m+1}(A)$ is not parallel rel F to $h_i(A)$ for $i = 1, \dots, m$ since then there would be an embedding $H: A \times I \longrightarrow M$ such that $H(\partial A \times I) \subseteq F$, $H(A \times \{0\}) \subseteq M_1$, $H(c_1 \times \{0\}) = \lambda$, $H(A \times \{1\}) \subseteq M - M_1$, and $HH^{-1}(\partial M_1)$ is an annulus from λ to a second component of ∂F_1 or contains an incompressible nonannular 2-submanifold of F_1. Either of the above is impossible. Thus $f_j(\partial A)$ does not lie entirely on F_1.

Let F_2 be the component of $F \cap \partial M_1$ on which $f_j(c_2)$ lies. As above either F_2 is homeomorphic to F_1 or $f_1(c_2)$ is freely homotopic in F_2 to a loop in ∂F_2. It is a consequence of Theorem 2 in [3] and Theorem 1′ in [1] that there are disjoint essential (in M) embeddings g_1, g_2, g_3: $(A, \partial A) \longrightarrow (M_1, \partial M_1)$ such that $g_1(c_1) \cup g_2(c_1) \cup g_3(c_1) = \partial F_1$. Since g_1 is essential and $g_1(A) \cap \bigcup_{i=1}^{n} h_i(A)$ is empty we may suppose that $g_1(A)$ and $h_1(A)$ are parallel rel F. Similarly $g_2(A)$ and $g_3(A)$ are parallel rel F to annuli in $\{h_i(A), i = 1, \dots, n\}$. It follows that $g_1(c_2) \cup g_2(c_2) \cup g_3(c_2) = \partial F_2$. Thus $S = F_1 \cup F_2 \cup g_1(A) \cup g_2(A) \cup g_3(A)$ and F_2 is homeomorphic to F_1.

It is a consequence of Theorem 2 in [3] and the existence of f_j: $(A, \partial A) \longrightarrow (M_1, \partial M_1)$ that there is a map $\bar{g}_1$: $(A, \partial A) \longrightarrow (M_1, F_1 \cup F_2)$ such that $\bar{g}_1(c_1) = g_1(c_1)$, $\bar{g}_1(c_2)$ is a component of ∂F_2, and $\bar{g}_1(\alpha)$ is homotopic rel its boundary to the product of an arc in F_1, $f_j(\alpha)$, and another arc in F_2. If $\bar{g}_1(c_2) \neq g_1(c_2)$, the collection $h_1, \dots, h_m$ can be extended using Theorem 1 in [1] which is impossible. Now if $\bar{g}_1(\alpha)$ is not homotopic rel its boundary to an arc in $g_1(A)$, there is an $(g_1(A))$-essential embedding $g_1^{\#}$: $(A, \partial A) \longrightarrow (M_1, \partial M_1)$ such that $g_1^{\#}(c_1) = g_1(c_1)$ and $g_1^{\#}(c_2) = g_1(c_2)$ as a consequence of Theorem 1′. We observe that if $g_1^{\#}$ is not essential in M, g_1 is not essential in M, a contradiction; and if we deform $g_1^{\#}(\partial A)$ slightly into $F_1 \cup F_2$, $g_1^{\#}(A)$ and $g_1(A)$ are not parallel rel F. Since $g_1^{\#}$: $(A, \partial A) \longrightarrow (M, F)$ would extend the collection $h_1, \dots, h_m$, this is impossible so $f_j(\alpha)$ is homotopic rel its boundary to the product of an arc in F_1 followed by a simple arc in $g_1(A)$, followed by an arc in F_2.

We assume that $f_j(\alpha) \subseteq \partial M_1$ as has been shown to be possible above and observe that f_j: $(A, \partial A) \longrightarrow (M_1, \partial M_1)$ induces a map $f: \mathcal{D} \longrightarrow M$ such that $f(\partial \mathcal{D}) = f_j(\partial A \cup \alpha)$. If $f(\partial \mathcal{D})$ is nullhomotopic in ∂M_1, there is a map f_j': $(A, \partial A) \longrightarrow (M_1, \partial M_1)$ such that $f_j'(A) \subseteq \partial M_1$ and $f_j | \partial A = f_j' | \partial A$. It would follow that $f_j(c_1)$ is homotopic to a loop in ∂F_1.

We can now construct a disk $\mathcal{D}_1$ properly embedded in M_1 such that $\mathcal{D}_1 \cap F_i$ is a simple arc properly embedded in F_i for $i = 1, 2$ and $\partial \mathcal{D}_1$ is not nullhomotopic in ∂M_1. Let N_3 be a regular neighborhood of $\mathcal{D}_1 \cup g_1(A)$. Now N_3 is a product line bundle and $f_j(c_i)$ is homotopic in F_i to a loop in $g_1(c_i) \cup (\mathcal{D}_1 \cap F_i)$ for $i = 1, 2$. Using an argument similar to the one above, we can find disks $\mathcal{D}_2$ and $\mathcal{D}_3$

properly embedded in M_1 such that $\mathcal{D}_2 \cap F_i$ ($\mathcal{D}_3 \cap F_i$) is an arc properly embedded in F_i for $i = 1, 2$, $\mathcal{D}_2 \cap \mathrm{Fr}(M_1) \subseteq g_2(A)$ ($\mathcal{D}_3 \cap \mathrm{Fr}(M_1) \subseteq g_3(A)$), and $\partial\mathcal{D}_2$ and $\partial\mathcal{D}_3$ are essential loops in ∂M_1. We may suppose that $(\mathcal{D}_1 \cap \mathcal{D}_2) \cup (\mathcal{D}_2 \cap \mathcal{D}_3) \cup (\mathcal{D}_1 \cap \mathcal{D}_3)$ is the union of a collection of disjoint simple arcs; and since M_1 is irreducible, it can be seen that M_1 is homeomorphic to $F_1 \times I$. This establishes claim (a). Proposition 4 now follows since $M_1 \cup N_2$ is a line bundle.

LEMMA 5. *Let M be a closed, irreducible, orientable 3-manifold and F a closed, connected, 2-sided, separating, incompressible surface embedded in M. If M admits an essential map f: $T \longrightarrow M$, then either*

(1) *M admits an essential embedding g: $T \longrightarrow M$.*

(2) *The closures M_1 and M_2 of the components of $M - F$ are twisted line bundles.*

PROOF. We may suppose after the usual argument that $f^{-1}(F)$ is the union of a collection of disjoint simple loops. Since $\pi_2(M) = 0$ and F is incompressible, we may assume that no simple loop in $f^{-1}(F)$ is inessential in T. We suppose that f has been chosen so that f is not homotopic to a map $\bar{f}$: $T \longrightarrow M$ such that $\bar{f}^{-1}(F)$ contains fewer loops than $f^{-1}(F)$. If $f^{-1}(F)$ is empty, Lemma 5 is an immediate consequence of Theorem 8 in [2] since $\pi_1(M_j) \longrightarrow \pi_1(M)$ is monic for $j = 1, 2$.

Let $\bar{R}_1$ be a regular neighborhood of $f(T) \cap F$ in F. Let R_1 be the smallest 2-submanifold of F such that $R_1 \supseteq \bar{R}_1$ and $\pi_1(R_1) \longrightarrow \pi_1(F)$ is monic. We suppose that f has been chosen so that the Euler characteristic of R_1 will be maximal and so that R_1 has a maximal number of components. Let $A_1, \ldots, A_n$ be the closures of the components of $T - f^{-1}(M_2)$ and $f_1, \ldots, f_n$ be the restriction of f to $A_1, \ldots, A_n$ respectively. It is a consequence of Proposition 3.3 in [2] that f_i: $(A_i, \partial A_i) \longrightarrow (M_1, F)$ is an essential map for $i = 1, \ldots, n$. It follows from Proposition 4 that there is a line bundle N_1 embedded in M_1 and maps $\bar{f}_i$: $(A_i, \partial A_i) \longrightarrow (M_1, F)$ homotopic to f_i for $i = 1, \ldots, n$ such that $\bigcup_{i=1}^{n} \bar{f}_i(\partial A_i) \subseteq N_1 \cap F$ where $N_1 \cap F$ is an incompressible surface embedded in R_1. If $N_1 \cap F$ is not a deformation retract of R_1, either R_1 has fewer components than $N_1 \cap F$ or the Euler characteristic of $N_1 \cap F$ is greater than that of N_1. Either of the above contradicts our choice of f.

A similar argument shows that there is a line bundle N_2 embedded in M_2 such that $N_2 \cap F$ is a deformation retract of R_1. Thus we may suppose that $N_1 \cap F = N_2 \cap F$. Note that if $N_1 \cap F = F$, $\partial N_1 = F$ is connected and both N_1 and N_2 are twisted line bundles.

Otherwise we observe that Proposition 4 guarantees that fibres of N_1 and N_2 are not homotopic rel their boundaries to arcs in F. Note that $N_1 \cup N_2$ is a 3-submanifold $\bar{M}$ of M. We claim that $\partial\bar{M}$, which is the union of a collection of tori is incompressible in M. If not, there is a disk $\mathcal{D}$ embedded in M such that $\mathcal{D} \cap \overline{M} = \partial\mathcal{D}$. We may suppose that $\mathcal{D} \cap F$ is the union of a collection of disjoint simple arcs. We suppose that $\mathcal{D}$ has been chosen so that $\mathcal{D} \cap F$ contains a minimal number of arcs. Let $\mathcal{D}_1$ be a component of $\mathcal{D} \cap M_1$ or $\mathcal{D} \cap M_2$ such that $\partial\mathcal{D} \cap \mathcal{D}_1$ is connected. If $\mathcal{D} = \mathcal{D}_1$, $\partial\mathcal{D}$ is freely homotopic in $\partial\overline{M}$ to a loop on F and $N_1 \cap F$ is not incompressible. Otherwise we suppose that $\partial\mathcal{D}_1 \cap N_1$ is not empty. If the arc $\partial\mathcal{D}_1 \cap N_1$ is not a spanning arc of the annular component of $\mathrm{Fr}(N_1)$ on which it lies, the number of arcs in $\mathcal{D} \cap F$ could be reduced. Otherwise a fibre of N_1 is homotopic rel its boundary across $\mathcal{D}_1$ to an arc in F. It follows that $\partial\overline{M}$ is incompressible in M and each component of $\partial\overline{M}$ is an essential torus in M. This completes the proof of Lemma 5.

REMARK 2. It is clear from the proof of Lemma 5 that if $\pi_1(f(T) \cap F) \longrightarrow \pi_1(F)$ is not onto, M admits an essential embedding $g\colon T \longrightarrow M$.

IV. The principal theorem.

THEOREM 1. *Let M be a closed, connected, irreducible orientable 3-manifold. Let $f\colon T \longrightarrow M$ be an essential map. Let F be an incompressible surface embedded in M. Then either M admits an essential embedding $g\colon T \longrightarrow M$ or M has a finite sheeted covering space homeomorphic to $F \times S^1$. Furthermore if M does not admit an essential embedding of T, M is either a bundle with base S^1 and fibre F or the union of two twisted line bundles each bounded by F.*

PROOF. We assume that F has minimal genus. If F does not separate M, Theorem 1 is an immediate consequence of Lemma 3. Otherwise we may assume that M does not admit an essential embedding of a torus and by Lemma 5 the closures M_1 and M_2 of the components of $M - F$ will be twisted line bundles. Let σ_j be an element of $\pi_1(M_j)$ so that $\sigma_j \notin \mathrm{im}(\pi_1(F) \longrightarrow \pi_1(M_j))$ for $j = 1, 2$ and $(\widetilde{M}, p)$ the covering space of M associated with the subgroup G of $\pi_1(M)$ generated by $\pi_1(F)$ and $\sigma_1\sigma_2$. Since σ_1 and G generate $\pi_1(M)$ and $\pi_1(M)/\pi_1(F) \cong Z$, $\widetilde{M}$ is a two-sheeted cover of M. It can be seen that neither component of $p^{-1}(F)$ separates $\widetilde{M}$. If $\widetilde{M}$ fails to admit an essential embedding of a torus, it is a consequence of Lemma 3 that $\widetilde{M}$ has a finite sheeted cover homeomorphic to $F \times S^1$.

Otherwise let $g\colon T \longrightarrow \widetilde{M}$ be an essential embedding. Let $\rho\colon \widetilde{M} \longrightarrow \widetilde{M}$ be the nontrivial covering translation. We may suppose that $g(T) \cap p^{-1}(F)$ is the union of a collection of disjoint, simple, essential loops since $g(T)$ and $p^{-1}(F)$ are incompressible in $\widetilde{M}$. We suppose further that $g(T)$ and $\rho g(T)$ are in general position and furthermore that if $\widetilde{F}$ is a component of $p^{-1}(F)$, one cannot decrease the number of points in $\widetilde{F} \cap g(T) \cap \rho g(T)$ via an isotopy $g(T)$ (and thus $\rho g(T)$). Now if $\pi_1(\rho g(T) \cap F) \longrightarrow \pi_1(F)$ is not onto (in particular if $pg(T) \cap F$ is not connected), it follows from Remark 2 that M admits an essential embedding. Otherwise by Remark 1, $\widetilde{M}$ (and thus M) has a finite sheeted cover homeomorphic to $F \times S^1$. This completes the proof of Theorem 1.

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